Finiteness properties of duals of local cohomology modules

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Abstract:
We investigate Matlis duals of local cohomology modules and prove that, in general, their zeroth Bass number with respect to the zero ideal is not finite. We also prove that, somewhat surprisingly, if we apply local cohomology again (i. e. to the Matlis dual of the local cohomology module), we get (under certain hypotheses) either zero or E, an R-injective hull of the residue field of the local ring R.

0 Introduction

Let I be an ideal of a noetherian local ring $(R, \mathfrak{m})$; $\mathrm{H}^i_I(R)$ is the i-th local cohomology module of R supported in I (see [Gr], [BS] for general facts on these modules) and $\mathrm{E} := \mathrm{E}_R(R/\mathfrak{m})$ a fixed R-injective hull of $R/\mathfrak{m}$. By D we denote the Matlis dual functor from $(R - mod)$ to $(R - mod)$, i. e.

$$D(M) = \mathrm{Hom}_R(M, \mathrm{E})$$

for every R-module M.

Recently there was some work on the modules $D(\mathrm{H}^i_I(R))$, see e. g. [HS1], [HS2] and [H1] – [H5]; one major motivation for the study of these modules is remark 1.1 below (originally [H5, Corollay 1.1.4] or [H1, section 0]) which connects regular sequences on $D(\mathrm{H}^i_I(R))$ in a clear way to the notion of a set-theoretic complete intersection ideal (i. e. an ideal which is, up to radical, generated by a regular sequence). Besides remark 1.1 various applications of results on $D(\mathrm{H}^i_I(R))$ are collected in [H5, section 6].

In some cases it is known (see e. g. [H5, theorem 3.1.3]) that the zero ideal of R (assume R is a domain) is associated to the R-module $D(\mathrm{H}^i_I(R))$; there is a conjecture (*) on the set of associated prime ideals of $D(\mathrm{H}^i_{\underline{x}}(R))$, where $\underline{x} = x_1, \ldots, x_i$ is a sequence in R: It is conjecture (*) from [H1] and [H5]:

$$(*) \qquad \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))) = \{\mathfrak{p} \in \mathrm{Spec}(R) |\, \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R/\mathfrak{p}) \neq 0\} \quad .$$

Clearly, if conjecture (*) holds, we have

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{\underline{x}}(R)))$$

provided $\mathrm{H}^i_{\underline{x}}(R) \neq 0$ holds. More details on conjecture (*) are contained in [H1] and [H5]. In any case, it is natural to ask for the associated Bass number, i. e. for the $Q(R)$-vector space dimension of

$$D(\mathrm{H}^i_I(R)) \otimes_R Q(R) \quad ,$$

where $Q(R)$ is the quotient field of R. Theorem 2.3, which is the first main result of this work, shows that this number is not finite, in general. In contrast to this, the results in section 3 show that $D(\mathrm{H}^i_I(R))$ is "small" in the following sense: $\mathrm{H}^i_I(D(\mathrm{H}^i_I(R)))$ is either E or zero; more precisely, it is E if I is a set-theoretic complete intersection ideal (theorem 3.1), while it is zero or E in a more general situation (theorem 3.2); note

(+) The author was partially supported by the DFG

that, for theorem 3.2, we use the so-called D-module structure on $\mathrm{H}^i_I(D(\mathrm{H}^i_I(R)))$, in particular we use [Ly, theorem 2.4] in connection with zero-dimensional D-modules and thus we have to assume equicharacteristic zero in the statement of theorem 3.2.

The author thanks Gennady Lyubeznik for helpful discussions, in particular on D-modules.

1 Prerequisites

This section collects some statements on Matlis duals of local cohomology modules that will be needed in the next two sections. For instance, the following is known about regular sequences on such Matlis duals:

1.1 Remark ([H5, Corollary 1.1.4])

Let $(R, \mathfrak{m})$ be a noetherian local ring, I a proper ideal of R, $h \in \mathbb{N}$ and $\underline{f} = f_1, \ldots, f_h \in I$ an R-regular sequence. The following statements are equivalent:

(i) $\sqrt{\underline{f}R} = \sqrt{I}$ (in this case I is a set-theoretic complete intersection).

(ii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is quasi-regular on $D(\mathrm{H}^h_I(R))$.

(iii) $\mathrm{H}^l_I(R) = 0$ for every $l > h$ and the sequence $\underline{f}$ is regular on $D(\mathrm{H}^h_I(R))$.

(The case $h = 0$ means

$$\begin{aligned}\sqrt{I} = \sqrt{0} &\iff \mathrm{H}^l_I(R) = 0 \text{ for every } l > 0 \\ &\iff \mathrm{H}^l_I(R) = 0 \text{ for every } l > 0 \text{ and } \Gamma_I(R) \neq 0 \quad).\end{aligned}$$

1.2 Remark

We want to calculate local cohomology modules and their Matlis duals in the following situation: k a field, $R = k[[X_1, \ldots, X_n]]$ a power series algebra over k in $n \in \mathbb{N}$ variables, $i \in \{0, \ldots, n\}$ and I the ideal $(X_1, \ldots, X_i)R$ of R, we want to calculate $\mathrm{H}^i_I(R)$ and $D(\mathrm{H}^i_I(R))$:

By considering the Čech-complex of R with respect to $X_1, \ldots, X_i$ it is not difficult to see that there is a canonical equality

$$\mathrm{H}^i_I(R) = k[[X_{i+1}, \ldots, X_n]][X_1^{-1}, \ldots, X_i^{-1}]$$

(by using this notation we mean, of course, the free $k[[X_{i+1}, \ldots, X_n]]$-module on the set of "inverse monomials" $\{X_1^{\nu_1} \cdot \ldots \cdot X_i^{\nu_i} | \nu_1, \ldots, \nu_i \leq 0\}$; this module has an obvious R-module structure). It is well-known and easy to see that one can realize $\mathrm{E} := \mathrm{E}_R(k)$ as $k[X_1^{-1}, \ldots, X_n^{-1}]$ (like before, this is the k-vector space with basis $\{X_1^{\nu_1} \cdot \ldots \cdot X_n^{\nu_n} | \nu_1, \ldots, \nu_n \leq 0\}$ having a natural R-module structure and containing $k = k \cdot 1 = k \cdot X_1^0 \cdot \ldots \cdot X_n^0$).

It is straight-forward to see that every element of

$$k[X_{i+1}^{-1}, \ldots, X_n^{-1}][[X_1, \ldots, X_i]]$$

(i. e. every formal power series in $X_1, \ldots, X_i$ and coefficients in $k[X_{i+1}^{-1}, \ldots, X_n^{-1}]$) determines (essentially by multiplication) an R-linear map

$$k[[X_{i+1}, \ldots, X_n]][X_1^{-1}, \ldots, X_i^{-1}] \to k[X_1^{-1}, \ldots, X_n^{-1}] \quad ,$$

i. e. an element of $D(\mathrm{H}^i_I(R))$. Furthermore, the element of $k[X_{i+1}^{-1}, \ldots, X_n^{-1}][[X_1, \ldots, X_i]]$ is uniquely determined by its associated map $\mathrm{H}^i_I(R) \to \mathrm{E}$; and finally, a tedious but easy calculation shows that every R-linear map $\mathrm{H}^i_I(R) \to \mathrm{E}$ arises this way, i. e. there is an equality

$$D(\mathrm{H}^i_I(R)) = k[X_{i+1}^{-1}, \ldots, X_n^{-1}][[X_1, \ldots, X_i]] \quad .$$

1.3 Remark (see [H5, subsection 7.2])
Let k be a field and $R = k[[X_1, \dots X_n]]$ a power series ring over k in n variables. Let

$$D(R,k) \subseteq \mathrm{End}_k(R)$$

be the (non-commutative) subring defined by the multiplication maps by $r \in R$ (for all $r \in R$) and by all k-linear derivation maps from R to R. $D(R,k)$ is the so-called ring of k-linear differential operators on R. [Bj] contains material on the ring $D(R,k)$ and on similar rings; $D(R,k)$-modules in relation with local cohomology modules have been studied in [Ly]. For $i = 1, \dots, n$ let ∂_i denote the partial derivation map from R to R with respect to X_i. Then, as an R-module, one has

$$D(R,k) = \oplus_{i_1,\dots,i_n \in \mathbb{N}} R \cdot \partial_1^{i_1} \dots \partial_n^{i_n} \quad . \tag{1}$$

Now, let $I \subseteq R$ be an ideal and $i \in \mathbb{N}$. We will demonstrate that there is a canonical left-$D(R,k)$-module structure on $D(\mathrm{H}^i_I(R))$. To do so, by identity (1), it is sufficient to determine the action of an arbitrary k-linear derivation $\delta : R \to R$ on $D(\mathrm{H}^i_I(R))$, to extend it to an action of $D(R,k)$ on $D(\mathrm{H}^i_I(R))$ and to show that this action is well-defined and satisfies all axioms of a left-$D(R,k)$-module. The derivation δ induces a k-linear map

$$R/I^v \to R/I^{v-1} \quad (v \geq 1)$$

and, in a canonical way, a map of complexes from the Čech complex of R/I^v with respect to $X_1, \dots, X_n$ to the Čech complex of R/I^{v-1} with respect to $X_1, \dots, X_n$ $(v \geq 1)$. By taking cohomology, we get a map

$$\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v) \to \mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^{v-1}) \quad (v \geq 1) \ ,$$

where $\mathfrak{m}$ stands for the maximal ideal of R. These maps induce a map

$$\mathrm{invlim}_{v \in \mathbb{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v)) \to \mathrm{invlim}_{v \in \mathbb{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v))$$

(note that the maps of the above inverse system are induced by the canonical epimorphisms $R/I^v \to R/I^{v-1}$). But, by local duality and $\mathrm{H}^i_I(R) = \mathrm{dirlim}_{v \in \mathbb{N}}(Ext^i_R(R/I^v, R))$, one has

$$\mathrm{invlim}_{v \in \mathbb{N}}(\mathrm{H}^{n-i}_{\mathfrak{m}}(R/I^v)) = D(\mathrm{dirlim}_{v \in \mathbb{N}}(Ext^i_R(R/I^v, R))) = D(\mathrm{H}^i_I(R)) \quad .$$

Now, having determined the action of the element δ on $D(\mathrm{H}^i_I(R))$, by (1) it is clear how to extend this to an action of $D(R,k)$ on $D(\mathrm{H}^i_I(R))$ such that $D(\mathrm{H}^i_I(R))$ becomes a left-$D(R,k)$-module (note that, for every k-linear derivation $\delta : R \to R$ and every $r \in R$, we have $\delta(r \cdot d) = \delta(r) \cdot d + r \cdot \delta(d)$, i. e. the action of $D(R,k)$ on $D(\mathrm{H}^i_I(R)$ makes it a left-$D(R,k)$-module). It is known (e. g. from various results in [H5, sections 2 – 4]) that, in general,

$$D(\mathrm{H}^i_I(R))$$

has infinitely many associated primes. On the other hand, one knows from [Ly, theorem 2.4 (c)] (at least if $\mathrm{char}(k) = 0$), that every finitely generated left-$D(R,k)$-module has only finitely many associated prime ideals (as R-module, of course). This shows that, in general, $D(\mathrm{H}^i_I(R))$ is an example of a non-finitely generated left-$D(R,k)$-module. In particular, $D(\mathrm{H}^i_I(R))$ is not holonomic in general (see [Bj] for the notion of holonomic modules).

2 The zeroth Bass number of $D(\mathrm{H}^i_I(R))$ (w. r. t. the zero ideal) is not finite in general

Let $(R, \mathfrak{m})$ be a noetherian local domain, $i \geq 1$ and $x_1, \ldots, x_i \in R$. Then one has

$$\{0\} \in \mathrm{Ass}_R(D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)))$$

in some situations (see e. g. [H5, theorem 3.1.3]); actually, if conjecture (*) (from [H1] and [H5]) holds, this is true provided $\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R) \neq 0$ holds. It is natural to ask for the associated Bass number of $D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$, i. e. the $Q(R)$-vector space dimension of

$$D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)) \otimes_R Q(R) \quad ,$$

where $Q(R)$ stands for the quotient field of R. As we will see below, this number is not finite in general; more precisely, we consider the following case: Let k be a field, $R = k[[X_1, \ldots, X_n]]$ a power series algebra over k in $n \geq 2$ variables, $1 \leq i < n$ and I the ideal $(X_1, \ldots, X_i)R$ of R; in this situation

$$\dim_{Q(R)}(D(\mathrm{H}^i_I(R)) \otimes_R Q(R)) = \infty$$

holds; this is the statement of theorem 2.3.

2.1 Remark

Note that in section 1.2 we introduced some notation on polynomials in "inverse variables" and we explained and proved the following formulas:

$$\mathrm{H}^i_I(R) = k[[X_{i+1}, \ldots, X_n]][X_1^{-1}, \ldots, X_i^{-1}] \quad ,$$

$$\mathrm{E}_R(k) = k[X_1^{-1}, \ldots, X_n^{-1}]$$

and

$$D(\mathrm{H}^i_I(R)) = k[X_{i+1}^{-1}, \ldots, X_n^{-1}][[X_1, \ldots, X_i]] \quad .$$

Also note that the latter module is different from (and larger) than the module

$$k[[X_1, \ldots, X_i]][X_{i+1}^{-1}, \ldots, X_n^{-1}] \quad .$$

2.2 Remark

The following proof of theorem 2.3, which says that a certain Bass number is infinite, is technical; its basic idea is the following one: Let k be a field, $R = k[[X, Y]]$ a power series algebra over k in two variables; then we have

$$\mathrm{H}^1_{XR}(R) = k[[Y]][X^{-1}]$$

and

$$D := D(\mathrm{H}^1_{XR}(R)) = k[Y^{-1}][[X]] \quad .$$

Set

$$\begin{aligned} d_2 &:= \sum_{l \in \mathbb{N}} Y^{-l^2} X^l \\ &= 1 + Y^{-1}X + Y^{-4}X^2 + Y^{-9}X^3 + \ldots \in D \end{aligned}$$

and let $r \in R \setminus \{0\}$ be arbitrary. Because of $r \neq 0$ we can write

$$r = X^{a+1} \cdot h + X^a \cdot g$$

with some $h \in R, g \in k[[Y]] \setminus \{0\}$. Then, at least for $l >> 0$, the coefficient of $r \cdot d_2$ in front of X^l is

$$h^* \cdot Y^{-(l-a-1)^2} + g \cdot Y^{-(l-a)^2}$$

for some $h^* \in k[[Y]]$. Now, if we write

$$g = c_b Y^b + c_{b+1} Y^{b+1} + \dots$$

for some $b \in \mathrm{N}, c_b \neq 0$ and observe the fact

$$-(l-a)^2 + b < -(l-a-1)^2 \quad (l >> 0) \quad ,$$

it follows that the term

$$c_b \cdot Y^{-(l-a)^2+b}$$

(coming from $h^* \cdot Y^{-(l-a-1)^2} + g \cdot Y^{-(l-a)^2}$) cannot be canceled out by any other term. In fact, for $l >> 0$, the lowest non-vanishing Y-exponent of the coefficient in front of X^l, is $-(l-a)^2 + b$. The crucial point is that the sequences $-(l-a)^2 + b$ and $-l^2$ agree up to the two shifts given by a and b. This means that some information about d_2 is stored in rd_2.

2.3 Theorem

Let k be a field, $R = k[[X_1, \dots, X_n]]$ a power series algebra over k in $n \geq 2$ variables, $1 \leq i < n$ and I the ideal $(X_1, \dots, X_i)R$ of R. Then

$$\dim_{Q(R)}(D(\mathrm{H}^i_I(R)) \otimes_R Q(R)) = \infty$$

holds.

Proof:

As the proof is technical we will first show the case $n = 2, i = 1$; in the remark after this proof we will explain how one can reduce the general to this special case. Set $X = X_1, Y = X_2$ and

$$D := D(\mathrm{H}^2_I(R)) = k[Y^{-1}][[X]] \quad .$$

For every $n \in \mathrm{N} \setminus \{0\}$, set

$$d_n := \sum_{l \in \mathrm{N}} Y^{-l^n} \cdot X^l \in D \quad .$$

It is sufficient to show the following statement: The elements $(d_n \otimes 1)_{n \in \mathrm{N} \setminus \{0\}}$ in $D \otimes_R Q(R)$ are $Q(R)$-linear independent:

We define an equivalence relation on $\mathbf{Z}^{\mathrm{N}}$ (the set of all maps from N to $\mathbf{Z}$, i. e. infinite sequences of integers) by saying that $(a_n), (b_n) \in \mathbf{Z}^{\mathrm{N}}$ are equivalent (short form: $(a_n) \sim (b_n)$) iff there exist $N, M \in \mathrm{N}$ and $p \in \mathbf{Z}$ such that

$$a_{N+1} = b_{M+1} + p, a_{N+2} = b_{M+2} + p, \dots$$

hold. It is easy to see that $\sim$ is an equivalence relation on $\mathbf{Z}^{\mathbb{N}}$. For every $d \in D$, we define $\delta(d) \in \mathbf{Z}^{\mathbb{N}}$ in the following way: Let $f_l \in k[Y^{-1}]$ be the coefficient of d in front of X^l; we set

$$(\delta(d))(l) := 0$$

if $f_l = 0$ and

$$(\delta(d))(l) := s$$

if s is the smallest Y-exponent of f_l, i. e.

$$f_l = c_s Y^s + c_{s+1} Y^{s+1} + \ldots + c_0 \cdot 1$$

for some $c_s \neq 0$.

Now suppose that $r_1, \ldots, r_{n_0} \in R$ are given such that $r_{n_0} \neq 0$. We claim that

$$\delta(r_1 d_1 + \cdot + r_{n_0} d_{n_0}) \sim \delta(d_{n_0})$$

holds. Note that if we prove this statement we are done, essentially because then $r_1 d_1 + \ldots + r_{n_0} d_{n_0}$ can not be zero.

It is obvious that one has $\delta(d + d') \sim \delta(d_{N_2})$ for given $d, d' \in D$ such that

$$\delta(d) \sim \delta(d_{N_1}), \delta(d') \sim \delta(d_{N_2}), N_2 > N_1 \quad .$$

For this reason it is even sufficient to prove the following statement: For a fixed $n \in \mathbb{N} \setminus \{0\}$ and for any $r \in R \setminus \{0\}$ one has

$$\delta(r d_n) \sim \delta(d_n) \quad .$$

We can write

$$r = X^{a+1} \cdot h + X^a \cdot g$$

with $a \in \mathbb{N}, h \in k[[X, Y]]$ and $g \in k[[Y]] \setminus \{0\}$. We get

$$\delta(r \cdot d_n) \sim \delta(\sum_{l \geq a+1} (h Y^{-(l-a-1)^n} + g Y^{-(l-a)^n}) X^l)$$

and we write

$$g = c_b Y^b + c_{b+1} Y^{b+1} + \ldots$$

with $c_b \in k^*$. Now, because of

$$-(l-a)^n + b < -(l-a-1)^n \quad (l >> 0)$$

it is clear that, for $l >> 0$, the smallest Y-exponent in front of X^l (of the power series $r \cdot d_n$) is $-(l-a)^n + b$. Therefore, one has

$$\delta(r \cdot d_n) \sim (-l^n) \sim \delta(d_n)$$

and we are done.

2.4 Remark
A proof of the general case of theorem 2.3 can be obtained e. g. in the following way: First, we use [H5, theorem 3.1.2] repeatedly to get a surjection

$$\mathrm{H}^i_{(X_1,\dots,X_i)R}(R) \to \mathrm{H}^{n-1}_{(X_1,\dots,X_{n-1})R}(R)$$

and hence an injection

$$D(\mathrm{H}^{n-1}_{(X_1,\dots,X_{n-1})R}(R)) \to D(\mathrm{H}^i_{(X_1,\dots,X_i)R}(R)) \quad ,$$

which allows us to reduce to the case $i = n-1$; then it is possible to adapt our proof of theorem 2.3 with some minor changes: Instead of working with maps $\mathbb{N} \to \mathbf{Z}$, one works with maps

$$\mathbb{N}^{n-1} \to \mathbf{Z}$$

and also with multi-indices instead of indices.

3 On the module $\mathrm{H}^i_I(D(\mathrm{H}^i_I(R)))$

In this section we assume $\mathrm{H}^l_I(R) = 0$ for every $l > \mathrm{height}(I)$ and investigate the local cohomology module

$$\mathrm{H}^{\mathrm{height}(I)}_I(D(\mathrm{H}^{\mathrm{height}(I)}_I(R))) \quad .$$

Our results say (essentially) that this module is $\mathrm{E}_R(R/\mathfrak{m})$ if I is a set-theoretic complete intersection (theorem 3.1) and that it is either $\mathrm{E}_R(R/\mathfrak{m})$ or zero in general (theorem 3.2):

3.1 Theorem
Let $(R,\mathfrak{m})$ be a noetherian local complete Cohen-Macaulay ring with coefficient field k and $x_1,\dots,x_i \in R$ ($i \geq 1$) a regular sequence in R. Set $I := (x_1,\dots,x_i)R$ (I is a set-theoretic complete intersection ideal of R). Then one has

$$\mathrm{H}^i_I(D(\mathrm{H}^i_I(R))) = \mathrm{E}_R(k) \quad .$$

Proof:
First we show a special case: Assume that $R = k[[X_1,\dots,X_n]]$ is a formal power series algebra over k in n variables and $x_1 = X_1,\dots,x_i = X_i$. Then, as we have seen in remark 1.3, we can write

$$\mathrm{H}^i_I(R) = k[[X_{i+1},\dots,X_n]][X_1^{-1},\dots,X_i^{-1}]$$

and

$$D(\mathrm{H}^i_I(R)) = k[X_{i+1}^{-1},\dots,X_n^{-1}][[X_1,\dots,X_i]] \quad .$$

As the functor H^i_I is right-exact, we have

$$\begin{aligned}
\mathrm{H}^i_I(D(\mathrm{H}^i_I(R))) &= \mathrm{H}^i_I(R) \otimes_R D(\mathrm{H}^i_I(R)) \\
&= k[[X_{i+1},\dots,X_n]][X_1^{-1},\dots,X_i^{-1}] \otimes_R k[X_{i+1}^{-1},\dots X_n^{-1}][[X_1,\dots,X_i]] \\
&\overset{(*)}{=} k[X_1^{-1},\dots,X_n^{-1}] \\
&= \mathrm{E}_R(k)
\end{aligned}$$

Proof of equality (*): The map

$$k[X_{i+1},\dots,X_n][X_1^{-1},\dots,X_i^{-1}] \otimes k[X_{i+1}^{-1},\dots X_n^{-1}][X_1,\dots,X_i] \to k[X_1^{-1},\dots,X_n^{-1}]$$

$$X_{i+1}^{r_{i+1}} \cdot \ldots \cdot X_n^{r_n} \cdot X_1^{-s_1} \cdot \ldots X_i^{-s_i} \otimes X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} \cdot X_1^{u_1} \cdot \ldots \cdot X_i^{u_i} \mapsto$$

$$\mapsto X_{i+1}^{r_{i+1}-t_{i+1}} \cdot X_n^{r_n-t_n} \cdot X_1^{u_1-s_1} \cdot \ldots \cdot X_i^{u_i-s_i} \text{ if } r_{i+1}-t_{i+1}, \ldots, r_n-t_n, u_1-s_1, \ldots, u_i-s_i \leq 0$$

and to zero otherwise, induces an R-linear map

(1) $$k[[X_{i+1}, \ldots, X_n]][X_1^{-1}, \ldots, X_i^{-1}] \otimes_R k[X_{i+1}^{-1}, \ldots X_n^{-1}][[X_1, \ldots, X_i]] \to k[X_1^{-1}, \ldots, X_n^{-1}] \quad ,$$

which is surjective and maps the k-vector space generating system

$$\{X_1^{-s_1} \cdot \ldots \cdot X_i^{-s_i} \otimes X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} | s_1, \ldots, s_i, t_{i+1}, \ldots, t_n \geq 0\}$$

of the vector space on the left side of (1) to the k-basis

$$\{X_1^{-s_1} \cdot \ldots \cdot X_i^{-s_i} \cdot X_{i+1}^{-t_{i+1}} \cdot \ldots \cdot X_n^{-t_n} | s_1, \ldots, s_i, t_{i+1}, \ldots, t_n \geq 0\}$$

of the vector space on the right side of (1), and therefore provides us with the desired isomorphism in our special case.
We come to the general case: Choose $x_{i+1}, \ldots, x_n \in R$ such that $\sqrt{(x_1, \ldots, x_n)R} = \mathfrak{m}$ ($x_1, \ldots x_n$ is a s. o. p. of R). Define

$$R_0 := k[[x_1, \ldots, x_n]] \subseteq R$$

R_0 is regular of dimension n and R is a finite-rank free R_0-module. Define $I_0 := (x_1, \ldots, x_i)R_0$. We have

$$\mathrm{H}_I^i(D(\mathrm{H}_I^i(R))) = \mathrm{H}_I^i(R) \otimes_R D(\mathrm{H}_I^i(R))$$

and

$$\mathrm{H}_I^i(R) = \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} R$$

and

$$\begin{aligned} D(\mathrm{H}_I^i(R))) &= \mathrm{Hom}_R(\mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} R, \mathrm{E}_R(k)) \\ &= \mathrm{Hom}_{R_0}(\mathrm{H}_{I_0}^i(R_0), \mathrm{E}_R(k)) \\ &\overset{(2)}{=} \mathrm{Hom}_{R_0}(\mathrm{H}_{I_0}^i(R_0), \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k))) \\ &= \mathrm{Hom}_{R_0}(R, D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \end{aligned}$$

For (2) we use the fact

$$\mathrm{E}_R(k) = \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k))$$

We get

$$\begin{aligned} \mathrm{H}_I^i(D(\mathrm{H}_I^i(R))) &= \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} \mathrm{Hom}_{R_0}(R, D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \\ &\overset{(3)}{=} \mathrm{Hom}_{R_0}(R, \mathrm{H}_{I_0}^i(R_0) \otimes_{R_0} D_{R_0}(\mathrm{H}_{I_0}^i(R_0))) \\ &= \mathrm{Hom}_{R_0}(R, \mathrm{E}_{R_0}(k)) \\ &\overset{(2)}{=} \mathrm{E}_R(k) \end{aligned}$$

For (3) we use the fact that R is a finite-rank free R_0-module.

3.2 Theorem

Let R be a noetherian local complete regular ring of equicharacteristic zero, $I \subseteq R$ an ideal of height $i \geq 1$, $x_1, \ldots, x_i \in I$ an R-regular sequence and assume that

$$\mathrm{H}^l_I(R) = 0 \text{ for every } l > i \quad .$$

Then $\mathrm{H}^i_I(D(\mathrm{H}^i_I(R)))$ is either $\mathrm{E}_R(k)$ or zero.
Proof:
We set

$$D := D(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(R))$$

By remark 1.1, we know that $x_1, \ldots, x_i$ is a D-regular sequence and, therefore, we have

$$\mathrm{H}^0_{(x_1,\ldots,x_i)R}(D) = \ldots = \mathrm{H}^{h-1}_{(x_1,\ldots,x_i)R}(D) = 0 \quad .$$

Because of this, an easy spectral sequence argument (applied to the composed functor $\Gamma_I \circ \Gamma_{(x_1,\ldots,x_i)R}$ and to the R-module D) shows that

$$\mathrm{H}^i_I(D) = \Gamma_I(\mathrm{H}^i_{(x_1,\ldots,x_i)R}(D)) \subseteq \mathrm{H}^i_{(x_1,\ldots,x_i)R}(D) = \mathrm{E}_R(k) \quad .$$

The last equality is theorem 3.1. But, from remark 1.3 and from [Ly, Example 2.1 (iv)], it is clear that

$$\mathrm{H}^i_I(R), D(\mathrm{H}^i_I(R)) \text{ and } \mathrm{H}^i_I(D(\mathrm{H}^i_I(R)))$$

all have a $D(R,k)$-module structure and so, from [Ly, theorem 2.4 (b)], we deduce that $\mathrm{H}^i_I(D)$ is either $\mathrm{E}_R(k)$ or zero. Furthermore, the natural injection

$$\mathrm{H}^i_I(R) \subseteq \mathrm{H}^i_{(x_1,\ldots,x_i)R}(R)$$

induces a surjection

$$D \to D(\mathrm{H}^i_I(R))$$

and hence, as H^i_I is right-exact, a surjection

$$\mathrm{H}^i_I(D) \to \mathrm{H}^i_I(D(\mathrm{H}^i_I(R))) \quad .$$

But again, the last module has a $D(R,k)$-module structure, and thus, from [Ly, theorem 2.4 (b)] and from what we know already, we conclude the statement.

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